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A Theory for Existence and Uniqueness of Solutions to Three Point Boundary Value Problems

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1. INTRODUCTION

In this paper we shall develop a technique for the existence and uniqueness of solutions to three point boundary value problems associated with the nth order differential equation

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}). \tag{1.1}$$

This technique arises out of a combination of George and Sutton [7] and Barr and Miletta [3], for the existence and uniqueness of solutions to (1.1) satisfying boundary conditions at two points. The technique arises naturally by combining the ideas of the above authors and by matching techniques developed by Dennis Barr and Tom Sherman [9], Murty, Rao, and Rao [10], and Johny Handerson [8] for third order differential equations. For a detailed discussion we refer [1–6] and [11–15].

2. PRELIMINARIES

In this paper we shall be concerned with the nth order differential equation

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}), \tag{2.1}$$

where f is a real valued function defined and continuous on $[a, b] \times R^n$. It will be assumed that solutions to initial value problems associated with (2.1) exist, are unique, and that solutions are defined throughout $[a, b]$. The boundary value problem is that of finding a solution ϕ of [2.1] on $[a, b]$ satisfying for $b > a$

$$y(a) = y_1, \qquad y(b) = y_2, \qquad y^{(i)}(a) = m_i \; (i = 1, 2, ..., n-2). \tag{2.2}$$

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The corresponding initial value problem is obtaining a solution ϕ of (2.1) satisfying

$$y(a) = y_i, \qquad y^{(i)}(a) = m_i \; (i = 1, 2, ..., n-1). \tag{2.3}$$

If ϕ_1 and ϕ_2 are solutions of (2.1) and (2.2) then by setting $\phi = \phi_1 - \phi_2$ we obtain

$$\phi^{(n)}(x) = F(x, \phi(x), \phi'(x), ..., \phi^{(n-1)}(x)) \tag{2.4}$$

$$\phi(a) = 0, \qquad \phi^{(i)}(a) = 0, \qquad \phi(b) = 0 \; (i = 1, 2, ..., n-2), \tag{2.5}$$

where

$$F(x, \phi(x), ..., \phi^{(n-1)}(x) = f(x, \phi_1(x), ..., \phi_1^{(n-1)}(x)) \\ - f(x, -\phi(x) + \phi_1(x), ..., -\phi^{(n-1)}(x) + \phi_1^{(n-1)}(x)).$$

Now $F(0, 0, ..., 0) = 0$ and $\phi(x) = 0$ is a solution of (2.4) satisfying (2.5). Thus we have the following result.

Result 2.1. The problem (2.4) and (2.5) has a solution ϕ if and only if $\phi + \phi_2$ is a solution of (2.1) and (2.2).

DEFINITION 2.1. A Liapunov function for (2.4) is a real valued function V defined on $D = [a, b] \times S$ where S is a closed subset of R^n and $(0, 0, ..., 0) \in S$ such that,

(i) $v(x, y_1, y_2, ..., y_n) = 0$ if $y_1 = 0$

(ii) $v(x, y_1, y_2, ..., y_n) > 0$ if $y_1 \neq 0$

(iii) $v(x, y_1, y_2, ..., y_n)$ is non-decreasing along the solution curves of (2.4).

By condition (iii) we mean that if ϕ is a solution of (2.4) then for $x_1 < x_2$, $v(x_1, \phi(x_1), \phi'(x_1), ..., \phi^{(n-1)}(x_1)) \leqslant v(x_2, \phi(x_2), \phi'(x_2), ..., \phi^{(n-1)}(x_2))$ with $(x_i, \phi(x_i), \phi'(x_i), ..., \phi^{(n-1)}(x_i)) \in D$.

LEMMA 2.1. *Let ϕ be a solution of* (2.1) *satisfying* (2.3) *and suppose that* $\phi^{(n-2)}(x) \neq 0$ *or* $\phi^{(n-1)}(x) \neq 0$ *on* $[a, b]$. *Then there exists an open interval* $I \subset [a, b]$ *such that* $\phi(x) \neq 0, \phi'(x) \neq 0, ..., \phi^{(n-1)}(x) \neq 0$ *on I.*

Proof. Suppose $\phi^{(n-1)}(x) \neq 0$ on $[a, b]$. Then by continuity of ϕ there exists an open interval $I_1 = (x_0, x_1) \subset [a, b]$ such that $\phi \neq 0$, $\phi \neq 0, ...,$ $\phi^{(n-2)} \neq 0$ on I_1 and either $\phi(x_0) = 0$ or $\phi'(x_0) = 0$ or $, ..., \phi^{(n-2)}(x_0) = 0$. Assume that $\phi^{(n-1)}(x) = 0$ on I_1. Then $\phi(x) = C_1 x^{(n-2)} + C_2 x^{(n-3)} + \cdots + C_{n-1}$ on I_1 where $C_1, C_2, ..., C_{n-1}$ are arbitrary constants. Since $\phi(a) =$

$\phi'(a) = \cdots = \phi^{(n-2)}(a) = 0$ and either $\phi(x_0) = 0$ or $\phi'(x_0) = 0, \ldots,$ or $\phi^{(n-2)}(x_0) = 0$ implies that $C_1 = C_2 = \cdots C_{n-1} = 0$, which contradicts the assumption. Since $\phi^{(n-1)}$ is continuous on $[a, b]$ there exists an open interval $I_2 \subset I_1$ where $\phi^{(n-1)} \neq 0$. In the case when $\phi^{(n-1)}(x) \neq 0$ on $[a, b]$ the result is immediate.

THEOREM 2.1. *Suppose V is a Liapunov function for* (2.4). *Then for any x_1 and x_2, $a \leqslant x_1 \leqslant x_2 \leqslant b$, there exists at most one solution of* (2.1) *satisfying* (2.2).

Proof. By the result 2.1, it suffices to show that $\phi = 0$ is the only solution of (2.4) and (2.5). Since $\phi(b) = 0$, it follows that $V(b, \phi(b), \phi'(b), \ldots, \phi^{(n-1)}(b)) = 0$. Hence by (ii) of Definition 2.1, ϕ cannot be non-zero for all $x \in [a, b]$. Hence $\phi = 0$ is the only solution of (2.4) and (2.5).

COROLLARY 2.1. *If there exists a Liapunov function as in Definition* 2.1 *except that* (ii) *holds when all $y_1, y_2, \ldots, y_n \neq 0$, then a solution of* (2.1) *and* (2.2) *whenever it exists is unique.*

3. A NECESSARY AND SUFFICIENT CONDITION

In this section we restrict our attention to the boundary value problem

$$x^{(n)} = F(t, x, x', \ldots, x^{(n-1)}) \tag{3.1}$$

$$x(a) = 0, \qquad x^{(i)}(a) = 0, \qquad x(\gamma) = 0 \ (i = 1, 2, \ldots, n-2), \tag{3.2}$$

where $F(t, 0, 0, \ldots, 0) = 0$ and $a \leqslant \gamma \leqslant b$. Thus we fix $t_1 = a$ and proceed to derive a necessary and sufficient condition for the existence and uniqueness of solutions of (3.1) and (3.2).

LEMMA 3.1. *Suppose $a < x_0 < b$ and $M > 0$. Let*

$$C^m\colon \{x\colon [a, b] \times R^n \to R \mid |x^{(n)}(t)| \leqslant m \quad \forall t \in [a, b];\ |x^{(n-1)}(a)| \leqslant m, \ldots, |x(a)| \leqslant m\}.$$

These exists a solution of (3.1) *satisfying* (3.2) *if and only if*

$$\operatorname*{Inf}_{x(t) \in C^m} \int_a^b |x^{(n)}(t) - f(t, x(t), x'(t), \ldots, x^{(n-1)}(t))| \, dt = 0.$$

Proof. Suppose the above infimum is zero. Let $\{x_k(t)\}$, $k=1, 2, \ldots$, be a sequence of functions in C^m such that

$$\lim_{k\to\infty} \int_a^b |x^{(n)}(t) - F(t, x_k(t), x_k'(t), \ldots, x_k^{(n-1)}(t))|\, dt = 0.$$

The sequence of functions $\{x_k^{(n-1)}(t)\}$, $k=1, 2, \ldots$, can easily be shown to be equicontinuous and uniformly bounded. For any t_1 and t_2

$$|x_k^{(n-1)}(t_1) - x_k^{(n-1)}(t_2)| \leqslant \int_{t_1}^{t_2} |x_k(t)|\, dt \leqslant m(t_2 - t_1).$$

Also

$$|x_k^{(n-1)}(t)| \leqslant x_k^{(n-1)}(a) + \int_a^t |x_k^{(n)}(t)\, dt \leqslant m + m(b-a).$$

Similarly the sequence of functions $\{x_k^{(n-2)}(t)\}$, $\{x_k^{(n-3)}(t)\}, \ldots, \{x_k(t)\}$ can easily be shown to be equicontinuous and uniformly bounded. Hence there exists a subsequence which we again call $\{x_k^{(n-1)}\}$, $\{x_k^{(n-2)}\}, \ldots, \{x_k\}$ converging uniformly to $x^{(n-1)}$, $x^{(n-2)}, \ldots, x$ on $[a, b]$. Since $x_k^{(j)}(a) = 0$ $(j=0, 1, \ldots, n-1)$ for all k, we have $x_a^{(j)} = 0$ $(j=0, 1, 2, \ldots, n-1)$ and $x_k(\gamma) = 0$ for all k, we have $x(\gamma) = 0$ and $x^{(n)}(t) = F(t, x(t), x'(t), \ldots, x^{(n-1)}(t))$.

Thus $x(t)$ is a solution of (3.1) satisfying (3.2). Conversely suppose $x(t)$ is a solution of (3.1) and (3.2), then the above infimum is obviously zero.

Note that in order to ensure the boundedness of F in Lemma 1 in [3] they have introduced an artificial compact set, D_M $(M>0)$, which actually does not serve our purpose. To run the proof of Lemma 1 in [3] we set D_M equal to

$$D_M = \{(t, \alpha, \beta)\colon |\alpha| \leqslant M \min[|t-\alpha|, |t-\beta|], 2\,|\beta| \leqslant M \text{ for all } t \in (a, b)\}.$$

For each $M>0$ and $(t, x_1, \ldots, x_n) \in [a, b] \times R^n$. Let $C^M(t, x_1, \ldots, x_n) = \{x \in C^M\colon x(t) = x_1, \ldots, x^{(n-1)}(t) = x_n\}$. We define a real valued function V_M with domain $[a, b] \times R^n$ as

$$V_M(t, x_1, \ldots, x_n) = \begin{cases} \{\displaystyle\operatorname*{Inf}_{x(t) \in C^M(t, x_1, \ldots, x_n)} \int_a^b |x^{(n)}(t) - F(t, x(t), \ldots, x^{(n-1)}(t))|\, dt\}, & x_1 \neq 0 \\ 0, \quad x = 0. \end{cases}$$

THEOREM 3.1. *Suppose there exists at most one solution to* (3.1) *and* (3.2) *for every γ, $a < \gamma < b$. Then for every $M>0$, V_M has the following properties*

(a) $V_M(t, x_1, x_2, ..., x_n) = 0$, $x_1 = 0$

(b) $V_M(t, x_1, x_2, ..., x_n) > 0$, $x_1 \neq 0$

(c) $V_M(t, x_1, x_2, ..., x_n)$ *is non-decreasing along the solution curves of* (3.1) *satisfying* $x(a) = 0$, $x^{(i)}(a) = 0$ $(i = 1, 2, ..., n-2)$, *and* $(t, x(t), x'(t), x''(t), ..., x^{(n-1)}(t)) \in C^M(t, x_1, ..., x_n)$ *for all* $t \in [a, b]$.

Proof. The proof is analogous to that of Theorem 3 in [3].

LEMMA 3.2. *Suppose* $b < x_0 < c$ *and* $M > 0$. *Let*

$$C^n = \{x\colon [b, c] \times R^n \to R \mid |x^n(t)| \leqslant M \quad \forall t \in [b, c],\ |x^{n-1}(b)| \leqslant M, ..., |x(b)| \leqslant M\}.$$

There exists a solution of (3.1) *satisfying*

$$x(b) = 0, \qquad x^{(i)}(b) = 0, \qquad x(c) = 0,\ \{i = 1, 2, ..., n-2\}.$$

Proof. The proof is analogous to the proof of Lemma 3.2 in [3].

THEOREM 3.2. *Let* $\lambda_1, \lambda_2, ..., \lambda_n, b \in R$, *with* $a < b < c$ *and suppose that*

(i) *for each* $m \in R$ *there exist solutions of* $y^{(n)} = f(x, y, y', ..., y^{(n-1)})$

$$y(a) = \lambda_1, \qquad y^{(j)}(b) = \lambda_{j+2}\ (j = 0, 1, ..., n-3),$$

$$y^{(n-i)}(b) = m \quad (i = 1, 2)$$

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)})$$

$$y^{(j)}(b) = \lambda_{j+2}\ (j = 0, 1, ..., n-3), \qquad y^{(n-i)}(b) = m, \qquad y(c) = \lambda_n.$$

(ii) *Suppose there exists a Liapunov function with properties* (a), (b), *and* (c) *of Theorem* 3.1. *Then there exists a unique solution of the three point boundary value problem*

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)})$$

$$y(a) = \lambda_1, \qquad y^{(j)}(b) = \lambda_{j+2}\ (j = 0, 1, ..., n-3), \qquad y(c) = \lambda_n.$$

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